

Two-Loop Effects and Current Status of the $^4\text{He}^+$ Lamb Shift

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Abstract: We report on recent progress in the treatment of two-loop binding corrections to the Lamb shift, with a special emphasis on S and P states. We use these and other results in order to infer an updated theoretical value of the Lamb shift in $^4\text{He}^+$.

PACS Nos.: 12.20.Ds, 31.30.Jv, 06.20.Jr, 31.15.-p

Résumé : Nous examinons le progrès récent concernant le traitement des corrections à l'ordre élevé des diagrammes à deux boucles contribuant au déplacement de Lamb, spécialement en ce qui concerne les états S et P . Par conséquent, on déduit de nouvelles précises valeurs théoriques pour le déplacement de Lamb en $^4\text{He}^+$.

1. Introduction

Recently, the higher-order two-loop corrections to the Lamb shift have been studied rather intensively, both within the $Z\alpha$ -expansion (see [1–8] and references therein) as well as within the nonperturbative (in $Z\alpha$) numerical approach, as described in Refs. [9–15]. In the current note, we review some recent progress for the so-called B_{60} coefficient, which is generated by the entire gauge-invariant set of two-loop diagrams, as depicted in Fig. 1. We also review some very recent progress [16] regarding the B_{60} coefficient for general excited hydrogenic states with nonvanishing angular momentum, with a special emphasis on states with P symmetry.

Applications of the recent progress to high-precision spectroscopy are numerous. As one example of current interest, the status of the $^4\text{He}^+$ Lamb shift ($1S$ and $2S$ states) is summarized, based on the recent analytic and numerical results, and on information about further known contributions to the Lamb shift from the literature (see in particular [7, 17, 18] and references therein).

2. Two-Loop Results

The two-loop energy shift of an atomic level in a hydrogenlike atomic system reads (in units with $\hbar = c = \epsilon_0 = 1$)

$$\Delta E_{\text{SE}}^{(2L)} = \left(\frac{\alpha}{\pi}\right)^2 \frac{(Z\alpha)^4 m_e}{n^3} H(Z\alpha), \quad (1)$$

where m_e is the electron mass, and H is a dimensionless function. In the current Section of this article, we are primarily concerned with recently obtained [8, 16] results for the normalized (or “weighted”)

Received xxx. Accepted xxx.

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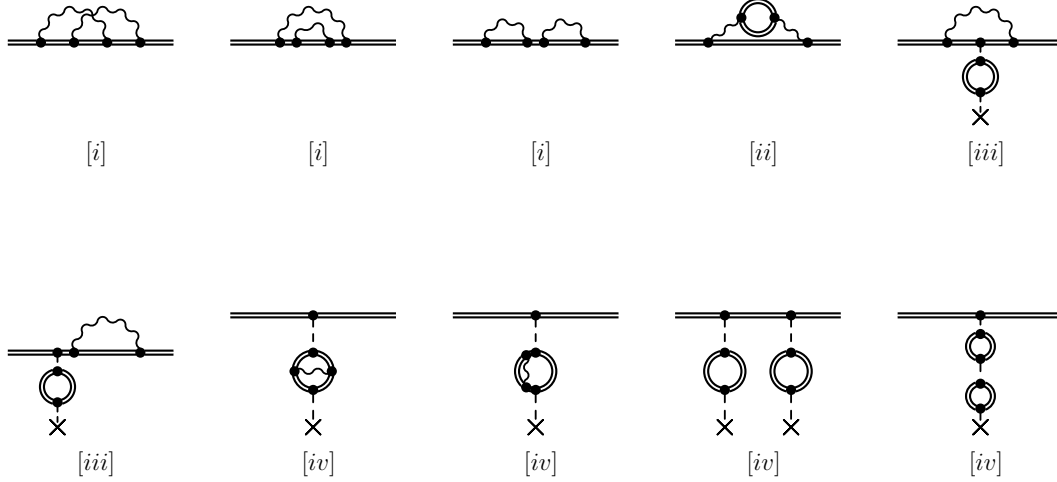


Fig. 1. Feynman diagrams for the two-loop self-energy corrections, separated into subsets i – iv according to Ref. [8]. Subset i is the pure two-loop self-energy, subset ii comprises the vacuum-polarization insertion into the virtual-photon line of the one-photon self-energy, subset iii contains vacuum-polarization corrections to the electron line in the one-photon self-energy, and subset iv contains remaining vacuum-polarization effects.

difference $H(nS, Z\alpha) - H(1S, Z\alpha)$ of S states, whose importance for the determination of fundamental constants has been stressed in Refs. [19–21], and for individual P states.

For these states and/or combinations of states, the first nonvanishing terms in the semi-analytic expansion of the dimensionless function $H(Z\alpha)$ in powers for $Z\alpha$ and $\ln(Z\alpha)$ read as follows,

$$H(Z\alpha) = B_{40} + (Z\alpha)^2 \left\{ B_{62} \ln^2[(Z\alpha)^{-2}] + B_{61} \ln[(Z\alpha)^{-2}] + B_{60} \right\}. \quad (2)$$

The first index of the B coefficients marks the power of $Z\alpha$, whereas the second corresponds to the power of the logarithm $\ln[(Z\alpha)^{-2}]$. For individual S states, we only mention here the existence of a B_{50} coefficient [1–3, 5], which goes beyond the coefficients listed in (2).

For the normalized difference of S states, we have [6]

$$B_{62}(nS) - B_{62}(1S) = \frac{16}{9} \left(\frac{3}{4} + \frac{1}{4n^2} - \frac{1}{n} + \gamma - \ln(n) + \Psi(n) \right), \quad (3)$$

where $\gamma = 0.577216 \dots$ is Euler's constant, and Ψ is the logarithmic derivative of the Gamma function. The normalized difference for B_{61} reads [6]

$$\begin{aligned} B_{61}(nS) - B_{61}(1S) &= \frac{4}{3} [N(nS) - N(1S)] \\ &+ \left[\frac{304}{135} - \frac{32}{9} \ln(2) \right] \left(\frac{3}{4} - \frac{1}{n} + \frac{1}{4n^2} + \gamma - \ln(n) + \Psi(n) \right). \end{aligned} \quad (4)$$

The normalized difference of the nonlogarithmic term can be expressed as [8]

$$B_{60}(nS) - B_{60}(1S) = b_L(nS) - b_L(1S) + A(n), \quad (5)$$

where $A(n)$ is an additional contribution beyond the n -dependence of the two-loop Bethe logarithm b_L . The result for A is [8, 22],

$$\begin{aligned}
A(n) = & \left(\frac{38}{45} - \frac{4}{3} \ln(2) \right) [N(nS) - N(1S)] - \frac{337043}{129600} - \frac{94261}{21600n} + \frac{902609}{129600n^2} \\
& + \left(\frac{4}{3} - \frac{16}{9n} + \frac{4}{9n^2} \right) \ln^2(2) + \left(-\frac{76}{45} + \frac{304}{135n} - \frac{76}{135n^2} \right) \ln(2) + \left(-\frac{53}{15} + \frac{35}{2n} - \frac{419}{30n^2} \right) \\
& \times \zeta(2) \ln(2) + \left(\frac{28003}{10800} - \frac{11}{2n} + \frac{31397}{10800n^2} \right) \zeta(2) + \left(\frac{53}{60} - \frac{35}{8n} + \frac{419}{120n^2} \right) \zeta(3) \\
& + \left(\frac{37793}{10800} + \frac{16}{9} \ln^2(2) - \frac{304}{135} \ln(2) + 8\zeta(2) \ln(2) - \frac{13}{3} \zeta(2) - 2\zeta(3) \right) [\gamma + \Psi(n) - \ln(n)]. \quad (6)
\end{aligned}$$

Here, $N(n)$ is a nonlogarithmic term generated by a Dirac- δ correction to a one-loop Bethe logarithm, as calculated in Ref. [23]. Of course, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function.

For P states, we have the known results [4, 8, 22]

$$\begin{aligned}
B_{62}(nP) &= \frac{4}{27} \frac{n^2 - 1}{n^2}, \\
B_{61}(nP_{1/2}) &= \frac{4}{3} N(nP) + \frac{n^2 - 1}{n^2} \left(\frac{166}{405} - \frac{8}{27} \ln 2 \right), \\
B_{61}(nP_{3/2}) &= \frac{4}{3} N(nP) + \frac{n^2 - 1}{n^2} \left(\frac{31}{405} - \frac{8}{27} \ln 2 \right). \quad (7)
\end{aligned}$$

The results for the nonlogarithmic terms of P can be inferred on the basis of Eq. (8.1) of Ref. [8] and the two-loop Bethe logarithms for P states (see [16] and Table 2),

$$\begin{aligned}
B_{60}(nP_{1/2}) &= b_L(nP) + \beta_4(nP_{1/2}) + \beta_5(nP_{1/2}) + \left[\frac{38}{45} - \frac{4}{3} \ln(2) \right] N(nP) - \frac{27517}{25920} - \frac{209}{288n} \\
&+ \frac{1223}{960n^2} + \frac{4}{27} \frac{n^2 - 1}{n^2} \ln^2(2) - \frac{38}{81} \frac{n^2 - 1}{n^2} \ln(2) + \left(\frac{25}{6} + \frac{3}{2n} - \frac{9}{2n^2} \right) \zeta(2) \ln(2) \\
&+ \left(-\frac{9151}{10800} - \frac{1}{4n} + \frac{1009}{1200n^2} \right) \zeta(2) + \left(-\frac{25}{24} - \frac{3}{8n} + \frac{9}{8n^2} \right) \zeta(3), \quad (8a)
\end{aligned}$$

$$\begin{aligned}
B_{60}(nP_{3/2}) &= b_L(nP) + \beta_4(nP_{3/2}) + \beta_5(nP_{3/2}) + \left[\frac{38}{45} - \frac{4}{3} \ln(2) \right] N(nP) - \frac{73321}{103680} + \frac{185}{1152n} \\
&+ \frac{8111}{25920n^2} + \frac{4}{27} \frac{n^2 - 1}{n^2} \ln^2(2) - \frac{11}{81} \frac{n^2 - 1}{n^2} \ln(2) + \left(\frac{299}{80} - \frac{3}{8n} - \frac{53}{20n^2} \right) \zeta(2) \ln(2) \\
&+ \left(-\frac{24377}{21600} + \frac{1}{16n} - \frac{3187}{3600n^2} \right) \zeta(2) + \left(-\frac{299}{320} + \frac{3}{32n} + \frac{53}{80n^2} \right) \zeta(3). \quad (8b)
\end{aligned}$$

In these formulas, β_4 and β_5 are low-energy spin-dependent contributions, defined in Eq. (4.21) of Ref. [8], whose numerical values may be inferred from one-loop calculations [23, 24].

The evaluation of the two-loop Bethe logarithm for $1S$ and $2S$ has been discussed in Ref. [25], and for $3S$ – $6S$ in Ref. [26]. For $1S$ and $2S$, there is no ambiguity in the definition of the Bethe logarithm,

Table 1. Total numerical values of the two-loop Bethe logarithms b_L for S and P states, broken down for the principal-value contribution \bar{b}_L and the squared-decay term $\delta^2 B_{60}$.

level	\bar{b}_L	$\delta^2 B_{60}$	b_L	level	\bar{b}_L	$\delta^2 B_{60}$	b_L
1S	-81.4(3)	0.0	-81.4(3)	—	—	—	—
2S	-66.6(3)	0.0	-66.6(3)	2P	-2.2(3)	-0.008	-2.2(3)
3S	-63.5(6)	-0.071	-63.6(6)	3P	-2.5(3)	-0.177	-2.7(3)
4S	-61.8(8)	-0.109	-61.9(8)	4P	-2.8(3)	-0.243	-3.0(3)
5S	-60.6(8)	-0.129	-60.7(8)	5P	-2.8(3)	-0.276	-3.1(3)
6S	-59.8(8)	-0.141	-59.9(8)	6P	-2.9(3)	-0.295	-3.2(3)

Table 2. Numerical values for the weighted difference of B_{60} for S states, and for individual P states.

level	$B_{60}(nS) - B_{60}(1S)$	level	$B_{60}(2P_{1/2})$	level	$B_{60}(2P_{3/2})$
2S	15.1(4)	2P _{1/2}	-1.6(3)	2P _{3/2}	-1.8(3)
3S	18.3(7)	3P _{1/2}	-2.0(3)	3P _{3/2}	-2.2(3)
4S	20.0(10)	4P _{1/2}	-2.4(3)	4P _{3/2}	-2.5(3)
5S	21.2(11)	5P _{1/2}	-2.4(3)	5P _{3/2}	-2.5(3)
6S	22.0(11)	6P _{1/2}	-2.5(3)	6P _{3/2}	-2.6(3)

which can roughly be explained as follows: essentially, the two-loop Bethe logarithm results from a renormalized integration over two photon energies. Both of these integrations are free of singularities for 1S and 2S. However, for all higher excited S states and all P states, one incurs real (rather than imaginary) contributions to the energy shift from the product of imaginary contributions due to singularities along both photon integrations (these are “squared decay rates” in the sense of Ref. [27]). It is thus necessary to make a clear distinction between the singularity-free, principal-value part \bar{b}_L and a real part $\delta^2 B_{60}$, which is incurred by “squared” (or, more precisely, products of) imaginary contributions from the pole terms. We write

$$b_L = \bar{b}_L + \delta^2 B_{60}, \quad (9)$$

where \bar{b}_L is obtained as the nonlogarithmic energy shift stemming from the nonrelativistic self-energy, with all integrations carried out by principal value, and $\delta^2 B_{60}$ is the corresponding contribution defined in Refs. [26, 27], due to squared imaginary parts. For 3S–6S states, the above separation is not really essential, because $\delta^2 B_{60}$ is a numerically marginal contribution as compared to \bar{b}_L (see Ref. [26]), and thus $b_L(nS) \approx \bar{b}_L(nS)$ to a very good approximation. For P states under investigation here, the distinction (9), surprisingly, is already important (see Table 1). Final numerical values of the weighted difference of B_{60} for S states, and for individual P states, are summarized in Table 2.

3. Status of the ${}^4\text{He}^+$ Lamb Shift

In the current section (we keep units with $\hbar = c = \epsilon_0 = 1$), we would like to use the results described above, in order to infer the current theoretical status of the Lamb shift of 1S and 2S in the ${}^4\text{He}^+$ ion. Before we start our actual discussion, however, we should remember that an ideal way to carry out a related calculation would involve a full-featured least-squares adjustment according to

Table 3. Contributions to the Lamb shifts of the $1S_{1/2}$ and $2S_{1/2}$ states in $^4\text{He}^+$. All equation numbers are connected to the contributions as listed in Ref. [7], unless indicated otherwise. SE= self-energy, VP= vacuum polarization, num. int.= numerical integration, m = mass of orbiting particle, for the electron $m = m_e$, and M = nuclear mass.

Order of contribution [$m_e c^2$]	Equation in Ref. [7]	$\mathcal{L}(1S)$ [MHz]	$\mathcal{L}(2S)$ [MHz]
$\alpha(Z\alpha)^4 \ln[(Z\alpha)^{-2}]$	Eq. (59) [part]	146 724.762	18 340.595
$\alpha(Z\alpha)^4$	Eq. (59) [part]	-40 796.296	-4 725.621
$\alpha^2(Z\alpha)^4$	Eq. (59) [part]	16.295	2.037
$\alpha^3(Z\alpha)^4$	Eq. (59) [part]	0.029	0.004
$\alpha(Z\alpha)^4$ (muonic vac. pol.)	Eq. (63)	-0.081	-0.010
$\alpha(Z\alpha)^4$ (hadronic vac. pol.)	Eq. (65)	-0.051	-0.006
$\alpha(Z\alpha)^5$ (SE+VP)	Eq. (73)	1 827.214	228.402
$\alpha^2(Z\alpha)^5$ (two one-loops)	Eqs. (74+76+79+80)	0.492	0.061
$\alpha^2(Z\alpha)^5$ (two-loop VP)	Eq. (75)	0.704	0.088
$\alpha^2(Z\alpha)^5$ (two-loop SE)	Eq. (81)	-10.709(1)	-1.339
$\alpha^2(Z\alpha)^5$ (two-loop sum)	sum of 3 above	-9.513(1)	-1.189
$\alpha(Z\alpha)^6 \ln^2[(Z\alpha)^{-2}]$	Eq. (84) [part]	-198.172	-24.771
$\alpha(Z\alpha)^6 \ln[(Z\alpha)^{-2}]$	Eq. (84) [part]	123.906	16.985
$\alpha(Z\alpha)^6 G_{\text{SE}}(Z\alpha)$	Ref. [28]	-82.542	-10.621
$\alpha(Z\alpha)^6 G_{\text{VP}}(Z\alpha)$	Ref. [29]	-1.685	-0.276
$\alpha(Z\alpha)^6 G_{\text{WK}}(Z\alpha)$	Eq. (101)	0.157	0.020
$\alpha^2(Z\alpha)^6 \ln^3[(Z\alpha)^{-2}]$	Ref. [4]	-1.153	-0.144
$\alpha^2(Z\alpha)^6 \ln^2[(Z\alpha)^{-2}]$	Ref. [6]	-0.213	0.037
$\alpha^2(Z\alpha)^6 \ln[(Z\alpha)^{-2}]$	Ref. [8]	2.666	0.279
$\alpha^2(Z\alpha)^6$	Refs. [8, 14]	-0.607(211)	-0.064(26)
$(Z\alpha)^5 m/M$	Eq. (136)	17.786	2.547
$(Z\alpha)^6 m/M$	Eq. (144)	-0.119	-0.015
$(Z\alpha)^7 \ln^2(Z\alpha) m/M$	Eq. (147)	-0.010	-0.001
$\alpha(Z\alpha)^5 m/M$	Eq. (151)+Eq. (46) of Ref. [30]	-0.112	-0.014
$\alpha(Z\alpha)^6 \ln^2[(Z\alpha)^{-2}] m/M$	Eq. (155)	0.018	0.002
$(Z\alpha)^4 (m/M)^2$	Eq. (15) of Ref. [31]	-0.053	-0.007
$Z (Z\alpha)^5 (m/M)^2$	approx., Eq. (152)	0.019(19)	0.002(2)
Nucl. size [rel., 1.680(5) fm]	Eq. (17) of this work	70.865(422)	8.860(53)
Nucl. size [rel., 1.673(1) fm]	Eq. (17) of this work	70.275(84)	8.786(11)
Sum [1.680(5) fm]		107693.112(472)	13837.031(59)
Sum [1.673(1) fm]		107692.522(228)	13836.957(29)

Ref. [33], which includes all available data from relevant high-precision experiments (see [17]) and which, in principle, allows for a deduction of the nuclear charge radius. In order to infer an approximate theoretical prediction, though, one has to use a charge radius obtained from other sources, and we

Table 4. Contributions to the Lamb shifts of the $2P_{1/2}$ and $2P_{3/2}$ states in $^4\text{He}^+$. As in Table 3, all equation numbers are connected to the contributions as listed in Ref. [7], unless indicated otherwise. The acronyms used for the corrections are also the same as in Table 3.

Order of contribution [$m_e c^2$]	Equation in Ref. [7]	$\mathcal{L}(2P_{1/2})$ [MHz]	$\mathcal{L}(2P_{3/2})$ [MHz]
$\alpha(Z\alpha)^4 \ln[(Z\alpha)^{-2}]$	Eq. (60) [part]	0.000	0.000
$\alpha(Z\alpha)^4$	Eq. (60) [part]	-206.095	200.725
$\alpha^2(Z\alpha)^4$	Eq. (60) [part]	0.414	-0.207
$\alpha^3(Z\alpha)^4$	Eq. (60) [part]	-0.003	0.002
$\alpha(Z\alpha)^4$ (muonic vac. pol.)	Ref. [17]	0.000	0.000
$\alpha(Z\alpha)^4$ (hadronic vac. pol.)	Ref. [17]	0.000	0.000
$\alpha(Z\alpha)^5$ (SE+VP)	Eq. (73)	0.000	0.000
$\alpha^2(Z\alpha)^5$ (two one-loops)	Eqs. (74+76+79+80)	0.000	0.000
$\alpha^2(Z\alpha)^5$ (two-loop VP)	Eq. (75)	0.000	0.000
$\alpha^2(Z\alpha)^5$ (two-loop SE)	Eq. (81)	0.000	0.000
$\alpha^2(Z\alpha)^5$ (two-loop sum)	sum of 3 above	0.000	0.000
$\alpha(Z\alpha)^6 \ln^2[(Z\alpha)^{-2}]$	Eq. (86) [part]	0.000	0.000
$\alpha(Z\alpha)^6 \ln[(Z\alpha)^{-2}]$	Eq. (86)	1.677	0.944
$\alpha(Z\alpha)^6 G_{\text{SE}}(Z\alpha)$	Ref. [28]	-0.329	-0.163
$\alpha(Z\alpha)^6 G_{\text{VP}}(Z\alpha)$	Ref. [29]	-0.022	-0.005
$\alpha(Z\alpha)^6 G_{\text{WK}}(Z\alpha)$	Eq. (101)	0.000	0.000
$\alpha^2(Z\alpha)^6 \ln^3[(Z\alpha)^{-2}]$	Ref. [4]	0.000	0.000
$\alpha^2(Z\alpha)^6 \ln^2[(Z\alpha)^{-2}]$	Ref. [4]	0.006	0.006
$\alpha^2(Z\alpha)^6 \ln[(Z\alpha)^{-2}]$	Ref. [8]	0.001	-0.001
$\alpha^2(Z\alpha)^6$	Refs. [8, 16]	-0.001	-0.001
$(Z\alpha)^5 m/M$	Eq. (136)	-0.138	-0.138
$(Z\alpha)^6 m/M$	Eq. (145)	0.007	0.007
$(Z\alpha)^7 \ln^2(Z\alpha) m/M$	Eq. (147)	0.000	0.000
$\alpha(Z\alpha)^5 m/M$	Eq. (151)	0.000	0.000
$\alpha(Z\alpha)^6 \ln^2[(Z\alpha)^{-2}] m/M$	Eq. (155)	0.000	0.000
$(Z\alpha)^4 (m/M)^2$	E.g., Ref. [32]	0.002	-0.001
$Z (Z\alpha)^5 (m/M)^2$	approx., Eq. (152)	0.000	0.000
Nucl. size	Eq. (17) of this work	0.000	0.000
Sum		-204.481	201.168

intend to follow this different route in the current work.

We partly base our evaluation on Refs. [7, 17, 18] and choose a format as in Table 1 of Ref. [17]. In the evaluations described in Tables 3 and 4, the 2002 CODATA values of the fundamental constants [34] were used.

For the Lamb shift \mathcal{L} , we use the implicit definition [31,32]

$$E = m_r [f(n, j) - 1] - \frac{m_r^2}{2(m_e + M)} [f(n, j) - 1]^2 + \mathcal{L} + E_{\text{hfs}}. \quad (10)$$

Here, E is the energy level of the bound two-body system under investigation, and $f(n, j)$ is the dimensionless Dirac energy. E.g., we have $f(1, 1/2) = f(1S) = \sqrt{1 - (Z\alpha)^2}$, and $f(2, 1/2) = f(2S) = \sqrt{\frac{1}{2}(1 + \sqrt{1 - (Z\alpha)^2})}$ for the $1S$ and $2S$ states, respectively. Furthermore, m_r is the reduced mass of the system, M is the nuclear mass, and E_{hfs} is the energy shift due to hyperfine effects, which are absent for the spinless ^4He nucleus.

In order to avoid confusion, we would like to include a few clarifying words regarding specific entries in Tables 3 and 4. In general, we have added the factor α to all scales for the contributions listed in Tables 3 and 4, giving all contributions with an overall scaling of $m_e c^2$, in contrast to $\alpha m_e c^2$, which had been used in Ref. [17]. Regarding the contribution of order $(Z\alpha)^4 \ln[(Z\alpha)^{-2}]$ in the first row of Table 3, it is worthwhile to note that this term represents the leading logarithm of the Lamb shift, given by

$$\alpha \frac{(Z\alpha)^4 m_e}{n^3} \ln[(Z\alpha)^{-2}] \left(\frac{m_r}{m_e} \right)^3. \quad (11)$$

Here, m_r is the reduced mass of the system, given by $m_r = m_e m_N / (m_e + m_N)$, where m_N is the mass of the nucleus. The reduced-mass dependence of the argument of the logarithm itself, $\ln[(Z\alpha)^{-2}] \rightarrow \ln[(Z\alpha)^{-2} m_e / m_r] = \ln[(Z\alpha)^{-2}] + \ln[m_e / m_r]$, is being included here and in Ref. [17] into the non-logarithmic term of order $\alpha(Z\alpha)^4$. One might wonder why there is a theoretical uncertainty associated to this contribution at all. The reason is that the most accurate theoretical value for the quantity (11) is obtained by expressing it in terms of the 2002 CODATA Rydberg constant, which has a relative uncertainty of 6.6×10^{-12} , and the 2002 CODATA fine-structure constant, which has a relative uncertainty of 3.3×10^{-12} . The latter is responsible for the small theoretical uncertainty of the leading logarithmic contribution to the Lamb shift of the $1S$ level. The term of order $\alpha(Z\alpha)^5$ contains both contributions from the self-energy and the vacuum polarization, as indicated by the explanatory note “SE+VP.”

The term of order $\alpha(Z\alpha)^6 \ln^2[(Z\alpha)^{-2}]$ corresponds to the self-energy coefficient A_{62} , as given e.g. in Ref. [35], and the indicated term of order $\alpha(Z\alpha)^6 \ln[(Z\alpha)^{-2}]$ is the sum of a self-energy and a vacuum-polarization contribution in this order. Note that the latter distinction differs from the one used in Ref. [17], where the term of order $\alpha(Z\alpha)^6 \ln^2[(Z\alpha)^{-2}]$ denotes the sum of a double logarithmic, and a single logarithmic *self-energy* contribution, and the term of order $\alpha(Z\alpha)^6 \ln[(Z\alpha)^{-2}]$ was reserved exclusively for the *vacuum-polarization* contribution in this order. The values of the self-energy remainder $G_{\text{SE}}(Z\alpha)$ for S and P states are listed in [28, 36]. The vacuum polarization remainder function $G_{\text{VP}}(Z\alpha)$ is taken from [29] and corresponds exclusively to the Uehling part of the one-loop vacuum polarization. It might be worthwhile to point out that at the current level of accuracy, it is entirely sufficient to consider the vacuum-polarization higher-order remainder for P states via the formula

$$\Delta E_{\text{VP}}(nP_j) = \frac{\alpha}{\pi} \frac{(Z\alpha)^6 m_e}{n^3} \left(\frac{m_r}{m_e} \right)^3 \{ A_{60}^{\text{VP}}(nP_j) + (Z\alpha) A_{70}^{\text{VP}}(nP_j) \}, \quad (12)$$

where the analytic coefficients read

$$A_{60}^{\text{VP}}(nP_{1/2}) = -\frac{3}{35} \frac{n^2 - 1}{n^2}, \quad A_{60}^{\text{VP}}(nP_{3/2}) = -\frac{2}{105} \frac{n^2 - 1}{n^2}, \quad (13a)$$

$$A_{70}^{\text{VP}}(nP_{1/2}) = \frac{41\pi}{2304} \frac{n^2 - 1}{n^2}, \quad A_{70}^{\text{VP}}(nP_{3/2}) = \frac{7\pi}{768} \frac{n^2 - 1}{n^2}. \quad (13b)$$

These have been obtained in Refs. [7, 16, 37] for general principal quantum number n .

For the B_{62} -term for S states of order $\alpha^2(Z\alpha)^6 \ln^2[(Z\alpha)^{-2}]$, the result from Ref. [6] was used, which supersedes the estimate given in Eq. (101) of Ref. [7]. For the analytic B_{61} -term of order $\alpha^2(Z\alpha) \ln[(Z\alpha)^{-2}]$, the result given in Ref. [8] provides the most recent value.

For the B_{60} coefficient corresponding to the nonlogarithmic term of order $\alpha^2(Z\alpha)^6$ for the ground state, two mutually contradictory results of $-61.6 \pm 15\%$ (Ref. [25]) and $-127 \pm 30\%$ (Ref. [14]) have been reported. The latter is based on an extrapolation of an all-order (in $Z\alpha$) numerical calculation. Note, however, that there is a known missing piece in the analytic result reported in Ref. [25], which is currently under study and which will need to be evaluated before final conclusions can be drawn. M. Eides [38] therefore suggested that a valid interim way of estimating the uncertainty of B_{60} would consist in taking the arithmetic mean of these two results, and taking the half difference as an estimate for the theoretical uncertainty. This uncertainty would comprise all higher-order analytic terms, as it involves a comparison to a nonperturbative (in $Z\alpha$) calculation. For the $2S$ state, we can use the result for $1S$ and add the weighted difference listed in Table 2. For P states, the results reported in Sec. 2 of this paper (see also [16]) provide enough information to eliminate all theoretical uncertainty at the current level of accuracy.

Finally, let us remark that a term of order $\alpha^2(Z\alpha)^7 \ln^2[(Z\alpha)^{-2}]$ could be estimated in principle on the basis of taking a “local” Lamb-shift potential that corresponds to the self-energy part of A_{50} , namely,

$$\delta V = 4\alpha(Z\alpha)^2 \left[\frac{139}{128} - \frac{1}{2} \ln(2) \right] \frac{\pi \delta^3(r)}{m_e^2}, \quad (14)$$

taken as an input for a Dirac- δ induced correction to the one-loop self-energy. The result of this approach could alternatively be used as an uncertainty estimate for all the higher-order terms. This procedure leads to the estimate

$$B_{72}(nS) = \pm \frac{8}{3} \pi \left[\frac{139}{128} - \frac{1}{2} \ln(2) \right], \quad (15)$$

For $1S$, this leads to a value of ± 0.337 MHz for the higher-order two-loop remainder. However, the uncertainty due to the B_{72} -contribution is already contained in the uncertainty of the remainder term of order $\alpha^2(Z\alpha)^6$, because in determining the uncertainty of B_{60} , a comparison was made to a non-perturbative numerical calculation for higher nuclear charge numbers. The latter necessarily contains all contributions from the B_{72} term and all higher-order remainders. The above result of ± 0.337 MHz therefore likely overestimates the uncertainty due to the higher-order two-loop remainder and is mentioned here only for illustrative purposes.

Concerning radiative-recoil corrections, we note that the discrepancy between [39] and [40] concerning radiative insertions into the electron line in the $\alpha(Z\alpha)^5 m/M$ radiative recoil correction was resolved in [30]. The analytical result from that work was used. According to Eq. (46) of Ref. [30], the coefficient multiplying the non-vacuum-polarization part of order correction of order $\alpha(Z\alpha)^5 m/M$ is $-1.32402796 \dots / n^3$.

The nuclear spin in ${}^4\text{He}^+$ is different as compared to atomic hydrogen. The former is a spin-1/2–spin-zero system, whereas the latter is a spin-1/2–spin-1/2 system. Recoil corrections of first order in the mass ratio are unaffected by the different spin of the nucleus as compared to hydrogen. However, recoil terms of order $Z(Z\alpha)^5(m/M)^2$ are nuclear spin-dependent. Without carrying out a detailed analysis, we approximately calculate the nuclear self-energy effects of order $Z(Z\alpha)^5(m/M)^2$ by leaving out the Pauli form-factor correction from Eq. (153) of Ref. [7], which is certainly absent for a spinless nucleus, and we conservatively take the Dirac form factor contribution as an uncertainty, while we note that a more detailed analysis would be of interest and currently lacking in the literature.

Finally, we add the nuclear-spin dependent correction listed in Eq. (15) of Ref. [31],

$$\Delta E = -\frac{1}{2} \frac{(Z\alpha)^4 m_e}{n^3} \left(\frac{m_e}{M}\right)^2 \delta_{l0}, \quad (16)$$

which is of second order in the mass ratio, for the spin-1/2–spin-zero system under investigation. This term is connected to the absence of the zitterbewegung term in the Breit Hamiltonian for a spinless nucleus. For P states, we also add terms of order $(Z\alpha)^4 \left(\frac{m_e}{M}\right)^2$, which do not depend on the zitterbewegung term and are given, e.g., in Ref. [32].

Concerning the nuclear-size correction, we would like to mention that a full integration of a nuclear potential with a fully relativistic wave function (e.g., within a hard-sphere approximation) turns out to be quite essential to obtain reliable values for this correction. We have carried out such an integration in the current investigation with the full Dirac wave functions and obtain results in agreement with Ref. [41]. Results obtained with two different values for the two different root-mean-square charge radii of 1.673(1) fm [42] and 1.680(5) fm [43] are given in Table 3. Note that the former charge radius of 1.673(1) fm has been questioned (see e.g. [44, 45]). The uncertainty due to the shape of the nuclear charge distribution can be estimated to be much smaller than the uncertainty due to the nuclear size, based on experience with highly charged ions [46]. For the ^4He -nucleus, a spherically symmetric model is well justified (closed shell, spin $I=0$).

The nuclear-size correction $\Delta E_{\text{fs}}(nS)$ and $\Delta E_{\text{fs}}(nP_j)$, for low nuclear charge numbers, can be approximated very well by the first few terms of an expansion in the two small parameters $Z\alpha$ and $m\langle r^2 \rangle^{1/2}$, with the result

$$\begin{aligned} \Delta E_{\text{fs}}(nS) = & \frac{(Z\alpha)^4 m^3 R^2}{n^3} \left[\frac{2}{5} - \frac{1}{3} (Z\alpha m R) + \right. \\ & \left. + (Z\alpha)^2 \left\{ -\frac{2}{5} \left[\ln \left(\frac{2 Z \alpha m R}{n} \right) + 2\gamma + \Psi(n) \right] + \frac{227}{150} + \frac{2}{5n} - \frac{9}{10n^2} \right\} \right], \quad (17a) \end{aligned}$$

$$\Delta E_{\text{fs}}(nP_j) = \frac{1}{10} \frac{(Z\alpha)^6 m^3 R^2}{n^3} \frac{n^2 - 1}{n^2} \delta_{j,1/2}. \quad (17b)$$

Here, $\gamma = 0.577216 \dots$ is Euler's constant, $\Psi(n)$ is the logarithmic derivative of the Gamma function, and R is the radius of the nucleus in a hard-sphere model, which is related to the root-mean-square radius $\langle r^2 \rangle^{1/2}$ by the following formula [see Eq. (7) of Ref. [41]],

$$R = \sqrt{\frac{5}{3}} \langle r^2 \rangle^{1/2} \sqrt{1 - \frac{3}{4} (Z\alpha)^2 \left\{ \frac{3}{25} \frac{\langle r^4 \rangle}{\langle r^2 \rangle^2} - \frac{1}{7} \right\}}. \quad (18)$$

For the $2P_{1/2}$ state, we obtain an upward finite nuclear-size energy shift of 353 Hz which is barely significant on the kHz level (see Table 4).

The results in Eqs. (17) and (18) have been obtained in the approximation of an infinitely heavy nucleus, and with exact Dirac wave functions for a point nucleus. Both of these approximations should be valid for $^4\text{He}^+$. In addition, it should be noted that both the results given in Eqs. (17) and (18) are in excellent numerical agreement with a full numerical integration of the finite-size potential with Dirac wave functions. The linear correction term in R , i.e. the term $-\frac{1}{3} (Z\alpha m R)$ in Eq. (17a), is a consequence of the exponential factor $\approx \exp(-Z\alpha m r/n)$ in the wave function, which should not be ignored in the evaluation of the finite-size effect, although this effect is primarily sensitive to the probability density at the origin (at the nucleus). Any further effects that influence the finite-size effect like nuclear polarization are here absorbed into the uncertainty of the nuclear radius (see, e.g., Ref. [47] for an illustrative discussion of some of the further aspects that are relevant to the finite-size effect).

4. Conclusions

In Sec. 2, we summarize recent theoretical results for the higher-order two-loop binding corrections to the Lamb shift. These results are used, in Sec. 3, to infer updated values for the Lamb shift of low-lying states of the $^4\text{He}^+$ ion. Some of the analytic coefficients used in the evaluation are given in Eqs. (13) and (17). The analytic expansion of the nuclear finite-size correction (17) might be useful in other contexts.

The recent progress in the field has allowed for an improvement of the theory beyond the limits set by the leading-order effects, and for some of the most accurate predictions in all of theoretical physics. In particular, we reemphasize that for P states, the results reported in Sec. 2 of this paper (see also [16]) provide sufficient information to eliminate all theoretical uncertainty at the kHz level for the Lamb shift in the $^4\text{He}^+$ ion.

Acknowledgments

Helpful conversations with Krzysztof Pachucki, Vladimir M. Shabaev, Vladimir A. Yerokhin and Peter J. Mohr are gratefully acknowledged. U.D.J. acknowledges support from the Deutsche Forschungsgemeinschaft (Heisenberg program).

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